

KNO scaling from a nearly Gaussian action for small- x gluons

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Transverse momentum integrated multiplicities in the central region of pp collisions at LHC energies satisfy Koba-Nielsen-Olesen scaling. We attempt to relate this finding to multiplicity distributions of soft gluons. KNO scaling emerges if the effective theory describing color charge fluctuations at a scale on the order of the saturation momentum is approximately Gaussian. From an evolution equation for quantum corrections which includes both saturation as well as fluctuations we find that evolution with the QCD β -function satisfies KNO scaling while fixed-coupling evolution does not. Thus, non-linear saturation effects and running-coupling evolution are both required in order to reproduce geometric scaling of the DIS cross section and KNO scaling of virtual dipoles in a hadron wave function.

I. INTRODUCTION

The color fields of hadrons boosted to the light cone are thought to grow very strong, parametrically of order $A^+ \sim 1/g$ where g is the coupling [1]. The fields of nuclei are enhanced further by the high density of valence charges per unit transverse area, which is proportional to the thickness $A^{1/3}$ of a nucleus [2].

In collisions of such strong color fields a large number of soft gluons is released. Due to the genuinely non-perturbative dynamics of the strong color fields a semi-hard “saturation scale” Q_s emerges; it corresponds to the transverse momentum where the phase space density of produced gluons is of order $1/\alpha_s$. The mean multiplicity per unit rapidity in high-energy collisions is then $\bar{n} \equiv \langle dN/dy \rangle \sim 1/\alpha_s$. Below we argue that a semi-classical effective theory of valence color charge fluctuations predicts that the variance of the multiplicity distribution is of order $k^{-1} \sim \mathcal{O}(\alpha_s^0)$ so that the perturbative expansion of \bar{n}/k begins at order $1/\alpha_s \gg 1$. We show that in the strong field limit then a Gaussian effective theory leads to Koba-Nielsen-Olesen (KNO) scaling [3]. This relates the emergence of KNO scaling in p_\perp -integrated multiplicity distributions from high-energy collisions to properties of soft gluons around the saturation scale.

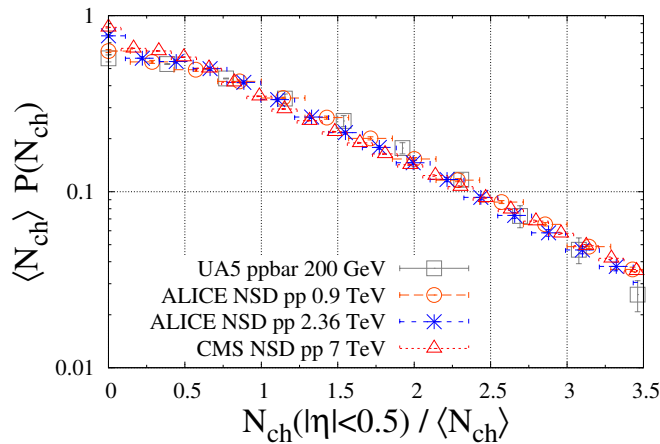


FIG. 1: KNO scaling of charged particle multiplicity distributions in non-single diffractive $pp / p\bar{p}$ collisions at various energies as measured by the UA5 [4], ALICE [5] and CMS [6] collaborations, respectively. Note that we restrict to the bulk of the distributions up to 3.5 times the mean multiplicity.

The KNO scaling conjecture refers to the fact that the particle multiplicity distribution in high-energy hadronic

collisions is *universal* (i.e., energy independent) if expressed in terms of the fractional multiplicity $z \equiv n/\bar{n}$. This is satisfied to a good approximation in the central (pseudo-) rapidity region at center of mass energies of 900 GeV and above [5, 6] as shown in fig. 1. On the other hand, UA5 data [4] taken at $\sqrt{s} = 200$ GeV appears to show a slightly distorted multiplicity distribution. This is in line with the observation that at lower energies higher-order factorial moments G_q of the distribution are energy dependent and significantly *different* from the reduced moments C_q [7]:

$$G_q \equiv \frac{\langle n(n-1)\cdots(n-q+1) \rangle}{\bar{n}^q} \quad , \quad C_q \equiv \frac{\langle n^q \rangle}{\bar{n}^q} \quad . \quad (1)$$

In fact, since the difference of G_q and C_q is subleading in the density of valence charges one may interpret this finding to indicate that the high density approximation is less accurate for $\sqrt{s} = 200$ GeV pp collisions. Approximate KNO scaling has been predicted to persist also for min-bias $p + Pb$ collisions (at LHC energies) in spite of additional Glauber fluctuations of the number of participants and binary collisions [8]. A more detailed discussion of multiplicity distributions at TeV energies is given in refs. [9], and references therein.

Transverse momentum integrated multiplicities in inelastic hadronic collisions are not governed by an external hard scale, unlike say multiplicity distributions in e^+e^- annihilation or in jets [10]. Hence, the explanation for the experimental observation should relate to properties of the distribution of produced gluons around the saturation scale Q_s .

II. KNO SCALING FROM A GAUSSIAN ACTION IN THE CLASSICAL LIMIT

We shall first discuss the multiplicity distribution of small- x gluons obtained from a Gaussian effective action for the color charge fluctuations of the valence charge densities ρ [2],

$$Z = \int \mathcal{D}\rho \, e^{-S_{MV}[\rho]} \quad , \quad (2)$$

$$S_{MV}[\rho] = \int d^2\mathbf{x}_\perp \int_{-\infty}^{\infty} dx^- \frac{\rho^a(x^-, \mathbf{x}_\perp) \rho^a(x^-, \mathbf{x}_\perp)}{2\mu^2(x^-)} \quad . \quad (3)$$

In the strong field limit a semi-classical approximation is appropriate and the soft gluon field (in covariant gauge) can be obtained in the Weizsäcker-Williams approximation as

$$A^+(z^-, \mathbf{x}_\perp) = -g \frac{1}{\nabla_\perp^2} \rho^a(z^-, \mathbf{x}_\perp) = g \int d^2\mathbf{z}_\perp \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} \frac{e^{i\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \mathbf{z}_\perp)}}{\mathbf{k}_\perp^2} \rho^a(z^-, \mathbf{z}_\perp) \quad . \quad (4)$$

Parametrically, the mean multiplicity obtained from the action (3) is then

$$\bar{n} \sim \frac{N_c(N_c^2 - 1)}{\alpha_s} Q_s^2 S_\perp \quad , \quad (5)$$

where S_\perp denotes a transverse area and $Q_s \sim g^2\mu$. The prefactor in (5) can be determined numerically [11, 12] but is not required for our present considerations.

One can similarly calculate the probability to produce q particles by considering fully connected diagrams with q valence sources ρ in the amplitude and q sources ρ^* in the conjugate amplitude (for both projectile and target, respectively). These can be expressed as [13]¹

$$\left\langle \frac{d^q N}{dy_1 \cdots dy_q} \right\rangle_{\text{conn.}} = C_q \left\langle \frac{dN}{dy_1} \right\rangle \cdots \left\langle \frac{dN}{dy_q} \right\rangle \quad , \quad (6)$$

where the reduced moments

$$C_q = \frac{(q-1)!}{k^{q-1}} \quad . \quad (7)$$

This expression is valid with logarithmic accuracy and was derived under the assumption that all transverse momentum integrals over $p_{T,1} \cdots p_{T,q}$ are effectively cut off in the infrared at a scale $\sim Q_s$ due to non-linear effects.

¹ The rapidities $y_1 \cdots y_q$ of the q particles should be similar. Here we assume that all particles are in the same rapidity bin.

The fluctuation parameter k in eq. (7) is of order

$$k \sim (N_c^2 - 1) Q_s^2 S_\perp . \quad (8)$$

Once again, the precise numerical prefactor (in the classical approximation) has been determined by a numerical computation to all orders in the valence charge density ρ [15].

The multiplicity distribution is therefore a negative binomial distribution (NBD) [13],

$$P(n) = \frac{\Gamma(k+n)}{\Gamma(k)\Gamma(n+1)} \frac{\bar{n}^n k^k}{(\bar{n}+k)^{n+k}} . \quad (9)$$

Indeed, multiplicity distributions observed in high-energy pp collisions (in the central region) can be described quite well by a NBD, see for example refs. [8, 14]. The parameter k^{-1} determines the variance of the distribution² and can be obtained from the (inclusive) double-gluon multiplicity:

$$\left\langle \frac{d^2 N}{dy_1 dy_2} \right\rangle_{\text{conn.}} = \frac{1}{k} \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle . \quad (10)$$

From this expression it is straightforward to see that the perturbative expansion of k^{-1} starts at $\mathcal{O}(\alpha_s^0)$ since the connected diagrams on the lhs of eq. (10) involve the same number of sources and vertices as the disconnected diagrams on the rhs of that equation (also see appendix). This observation is important since *in general* the NBD (9) exhibits KNO scaling only when $\bar{n}/k \gg 1$, and if k is not strongly energy dependent. A numerical analysis of the multiplicity distribution at 2360 GeV, for example, achieves a good fit to the data for $\bar{n}/k \simeq 6 - 7$ [8], which we confirm below. Such values for \bar{n}/k have also been found for peripheral collisions of heavy ions from ab initio solutions of the classical Yang-Mills equations [16]; furthermore those solutions predict that $\bar{n}/k < 1$ for central collisions of $A \sim 200$ nuclei.

To illustrate how deviations from KNO scaling arise it is instructive to consider a “deformed” theory with an additional contribution to the quadratic action. We shall add a quartic operator [17],

$$S_Q[\rho] = \int d^2 \mathbf{v}_\perp \int_{-\infty}^{\infty} dv_1^- \left\{ \frac{\rho^a(v_1^-, \mathbf{v}_\perp) \rho^a(v_1^-, \mathbf{v}_\perp)}{2\tilde{\mu}^2(v_1^-)} + \int_{-\infty}^{\infty} dv_2^- \frac{\rho^a(v_1^-, \mathbf{v}_\perp) \rho^a(v_1^-, \mathbf{v}_\perp) \rho^b(v_2^-, \mathbf{v}_\perp) \rho^b(v_2^-, \mathbf{v}_\perp)}{\kappa_4} \right\} . \quad (11)$$

We assume that the contribution from the quartic operator is a small perturbation since $\kappa_4 \sim g(gA^{1/3})^3$ while $\tilde{\mu}^2 \sim g(gA^{1/3})$. In the classical approximation the mean multiplicity is unaffected by the correction as it involves only two-point functions³. On the other hand, k^{-1} as defined in (10) now becomes

$$\frac{N_c^2 - 1}{2\pi} Q_s^2 S_\perp \frac{1}{k} = 1 - 3\beta (N_c^2 + 1) \quad \left(\text{with } \beta \equiv \frac{C_F^2}{6\pi^3} \frac{g^8}{Q_s^2 \kappa_4} \left[\int_{-\infty}^{\infty} dz^- \mu^4(z^-) \right]^2 \right) . \quad (12)$$

Therefore, in the classical approximation

$$\frac{\bar{n}}{k} \sim \frac{N_c}{\alpha_s} (1 - 3\beta (N_c^2 + 1)) . \quad (13)$$

This result illustrates that \bar{n}/k decreases as the contribution of the $\sim \rho^4$ operator increases. We repeat that the derivation assumed that the correction is small so that (13) does not apply for large values of βN_c^2 .

Ref. [17] estimated by entirely different considerations that for protons $\beta \simeq 0.01$ at $x = 10^{-2}$. That would correspond to a smaller value of \bar{n}/k by a factor of 1.43 than for the Gaussian theory. Assuming that RG flow with energy approaches a Gaussian action [18], \bar{n}/k should increase by about this factor. NBD fits to the data shown in fig. 2 confirm that \bar{n}/k indeed increases with energy, which might indicate flow towards a Gaussian action; however, the observed increase from $\sqrt{s} = 200$ GeV to 7 TeV is much stronger: a factor of about 3. This apparent discrepancy could be resolved at least partially by running of the coupling in eqs. (5,13) with Q_s but this requires more careful analysis⁴.

² More precisely, the width is given by $\bar{n} \sqrt{k^{-1} + \bar{n}^{-1}} \sim \bar{n}/\sqrt{k}$; the latter approximation applies in the limit $\bar{n}/k \gg 1$, see below.

³ The two-point functions $\langle \rho \rho \rangle$ in the theories (3) and (11) need to be matched. Thus, the “bare” parameters μ^2 in (3) and $\tilde{\mu}^2$ in (11) are *different* as the latter absorbs some self-energy corrections. We refer to ref. [17] for details.

⁴ In fact, the running of α_s at the effective scale Q_s is taken into account if the mean multiplicity is computed with energy evolved unintegrated gluon distributions like e.g. in refs. [8, 12].

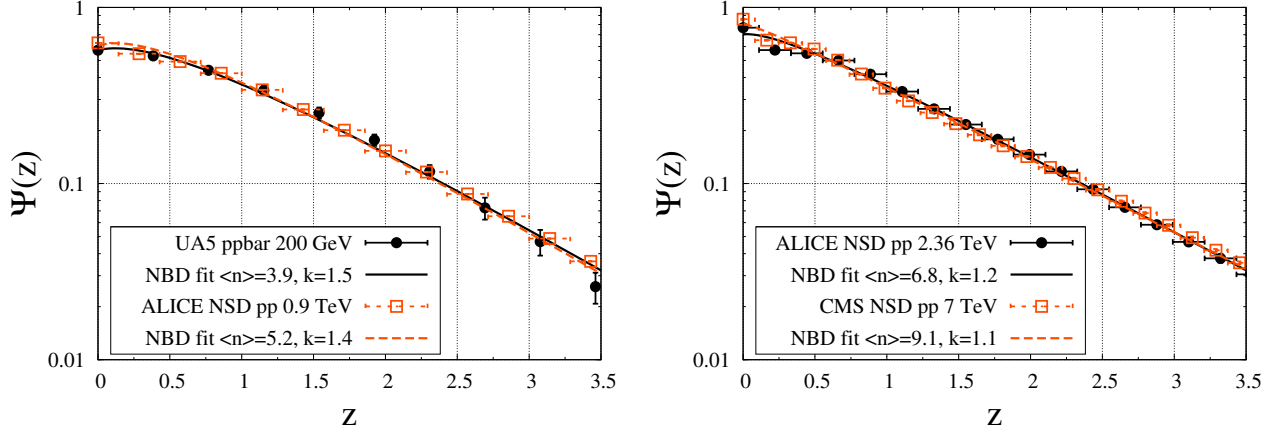


FIG. 2: KNO scaling plots of charged particle multiplicity distributions at $|\eta| < 0.5$ in NSD collisions at various energies and NBD fits; $z \equiv N_{\text{ch}}/\langle N_{\text{ch}} \rangle$ and $\Psi(z) \equiv \langle N_{\text{ch}} \rangle P(N_{\text{ch}})$. Note that the mean multiplicity quoted for the fits has been rescaled by 1.5 to include neutral particles; also, that here k is integrated over the transverse plane of the collision.

III. QUANTUM EVOLUTION AND THE DISTRIBUTION OF DIPOLES IN THE HADRONIC WAVE FUNCTION

In the previous section we considered the multiplicity distribution of “produced gluons” in a collision of classical YM fields sourced by classical color charges ρ moving on the light-cone. At high energies though (i.e., when $\alpha_s \log x^{-1} \sim 1$) the classical fields are modified by quantum fluctuations [1]. Resummation of boost-invariant quantum fluctuations leads to an energy dependent saturation scale, for example, as required in order to reproduce the growth of the multiplicity \bar{n} with energy. In particular, the energy dependence of the *mean* saturation scale, averaged over all “evolution ladders” (distribution of quantum emissions), can be obtained by solving the running-coupling BK equation [19].

Instead, in this section we shall solve a *stochastic* evolution equation which accounts both for saturation (non-linear) effects as well as for the fluctuations of the rapidities and transverse momenta of the virtual gluons in the wave function of a hadron before the collision. We do this in order to determine the multiplicity distribution (rather than just the mean number) of dipoles in a hadronic wave function boosted to rapidity Y .

We shall do so by solving via Monte-Carlo techniques the following evolution equation for $P[n(x), Y]$, which is the probability for the dipole size distribution $n(x)$ to occur:

$$\frac{\partial P[n(x), Y]}{\partial Y} = \int_z f_z[n(x) - \delta_{xz}] P[n(x) - \delta_{xz}, Y] - \int_z f_z[n(x)] P[n(x), Y]. \quad (14)$$

Note that in this section $x = \log 1/r^2$ denotes the logarithmic dipole size (conjugate to its transverse momentum) rather than to a light-cone momentum fraction. This equation has been studied before in ref. [20] for fixed α_s and in ref. [21] for running $\alpha_s(r^2)$. Those papers also provide references to related earlier work.

The first term in (14) is a gain term due to dipole splitting while the second term corresponds to loss due to “recombination”.

$$f_z[n(x)] = \frac{T_z[n(x)]}{\alpha(z)} \quad (15)$$

is the splitting rate and

$$T_z[n(x)] = 1 - \exp \int_x n(x) \log(1 - \tau(z|x)) \quad (16)$$

is the dipole scattering amplitude for a dipole projectile of size z to scatter off the target with the dipole distribution $n(x)$. Note that $T_z[n(x)]$ is non-linear in the dipole density as it involves also the *pair* (and higher) densities. Finally,

$$\tau(x|y) = \alpha(x)\alpha(y) \exp(-|x - y|) \equiv \alpha(r_{<}^2) \alpha(r_{>}^2) \frac{r_{<}^2}{r_{>}^2} \quad (17)$$

is the elementary dipole-dipole scattering amplitude at LO in perturbative QCD. For more details we refer to ref. [21]. Here, we recall only that it was found there that evolution with a running coupling suppresses fluctuations in the tails of the travelling waves and so restores approximate geometric scaling [22].

We have determined the multiplicity distribution of dipoles with size $\sim 1/Q_s(Y)$,

$$N_i(Y) = \int_{r^2 < 1/\langle Q_s^2(Y) \rangle} dx n_i(x, Y) \quad (i = 1 \cdots 10^5) \quad (18)$$

by evolving a given initial configuration $n(x, Y = 0)$ 10^5 times. Despite starting with a fixed initial condition, evolution introduces fluctuations in the rapidities where splittings occur, and in the sizes of the emerging dipoles.

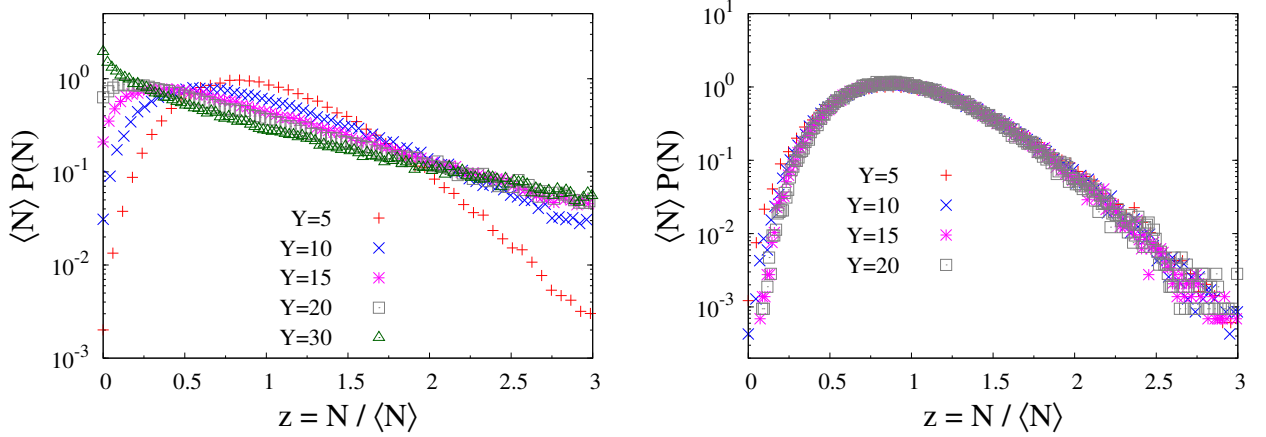


FIG. 3: Multiplicity distribution of virtual dipoles with size $r^2 < 1/\langle Q_s^2(Y) \rangle$ in the wave function. Left: evolution with trivial β -function, $\alpha_s = \text{const.}$ Right: QCD β -function.

In fig. 3 we show that fixed-coupling evolution does not obey KNO scaling of the distribution of virtual quanta while running-coupling evolution does. The shape of the distribution however looks different than the measured distribution of produced particles from fig. 1. This could be due to the fact that our evolution model does not treat diffusion in impact parameter space. Hence, $P(N)$ shown in fig. 3 should be interpreted as the multiplicity distribution at the center of the hadron.

Acknowledgments

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Appendix A: The moment $C_2 = k^{-1}$ with the quartic action

We can obtain the fluctuation parameter k by calculating the inclusive double gluon multiplicity and expressing it in terms of the single inclusive or mean multiplicity. The connected two particle production cross section for gluons with rapidity y_1 and y_2 has the form:

$$N_2(p, q) \equiv \left\langle \frac{d^2 N}{dy_1 dy_2} \right\rangle - \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle \equiv \left\langle \frac{d^2 N}{dy_1 dy_2} \right\rangle_{\text{conn.}} \quad (A1)$$

$\left\langle \frac{dN}{dy} \right\rangle$ is the mean multiplicity and the brackets denote an average over events. $N_2(p, q)$ is given by:

$$N_2(p, q) = \frac{g^{12}}{4(2\pi)^6} f_{gaa'} f_{g'bb'} f_{gcc'} f_{g'dd'} \int \prod_{i=1}^4 \frac{d^2 k_i}{(2\pi)^2 k_i^2} \frac{L_\mu(p, k_1) L^\mu(p, k_2) L_\nu(q, k_3) L^\nu(q, k_4)}{(p - k_1)^2 (p - k_2)^2 (q - k_3)^2 (q - k_4)^2} \times \\ \langle \rho_1^{*a}(k_2) \rho_1^{*b}(k_4) \rho_1^c(k_1) \rho_1^d(k_3) \rangle \langle \rho_2^{*a'}(p - k_2) \rho_2^{*b'}(q - k_4) \rho_2^{c'}(p - k_1) \rho_2^{d'}(q - k_3) \rangle .$$

L^μ denotes the Lipatov vertex, for which:

$$L_\mu(p, k)L^\mu(p, k) = -\frac{4k^2}{p^2}(p - k)^2. \quad (\text{A2})$$

For the four-point function in the target and projectile fields we use [17]

$$\begin{aligned} & \langle \rho_2^{*a'}(p - k_2)\rho_2^{*b'}(q - k_4)\rho_2^{c'}(p - k_1)\rho_2^{d'}(q - k_3) \rangle = \\ & (2\pi)^4 \left[\int dz^- \tilde{\mu}^2(z^-) \right]^2 \left[\delta^{a'b'}\delta^{c'd'}\delta(p + q - k_2 - k_4)\delta(p + q - k_1 - k_3) \right. \\ & \quad \left. + \delta^{a'c'}\delta^{b'd'}\delta(k_1 - k_2)\delta(k_3 - k_4) + \delta^{a'd'}\delta^{b'c'}\delta(p - q - k_2 + k_3)\delta(p - q - k_1 + k_4) \right] \\ & - (2\pi)^4 \frac{2}{\pi^2 \kappa_4} \left[\int dz^- \tilde{\mu}^4(z^-) \right]^2 \left(\delta^{a'b'}\delta^{c'd'} + \delta^{a'c'}\delta^{b'd'} + \delta^{a'd'}\delta^{b'c'} \right) \delta(k_1 + k_3 - k_2 - k_4). \end{aligned}$$

The first two lines on the rhs of the above equation originate from the quadratic part of the action while the third line is due to the quartic operator. The product of the Gaussian parts of the two four-point functions gives nine terms, one of which ($\sim \delta^{ac}\delta^{bd}\delta^{a'c'}\delta^{b'd'}$) corresponds to a disconnected contribution. It exactly cancels the second term in eq. (A1).

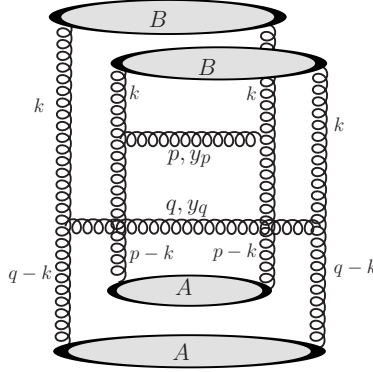


FIG. 4: One of eight connected diagrams for two-gluon production with the quadratic MV action.

Four of the other eight terms ($\sim \delta^{ac}\delta^{bd}$ or $\sim \delta^{a'c'}\delta^{b'd'}$) give identical leading contributions to double gluon production. They correspond to a “rainbow” diagram like the one shown in Fig. 4. In the “rainbow” diagram, on one side (target or projectile), the ρ ’s corresponding to the same gluon momentum are contracted with each other. The remaining four “non-rainbow” diagrams are suppressed relative to the terms we keep at large p and q [13]. Hence, the leading Gaussian contribution is:

$$\sim \frac{g^{12}}{8\pi^7} \left[\int dz^- \tilde{\mu}^2(z^-) \right]^4 \frac{S_\perp}{Q_s^2} \frac{N_c^2(N_c^2 - 1)}{p^4 q^4}. \quad (\text{A3})$$

The same reasoning applies also for the additional quartic contribution and only “rainbow” diagrams are considered, like the one in Fig. 5. There are two of them (one for the projectile and one for the target) to first order in κ_4^{-1} , and their contribution is:

$$\begin{aligned} & \sim -\frac{g^{12}}{2(2\pi)^6} f_{ga'a'} f_{g'b'b'} f_{g'c'c'} f_{g'd'd'} \int \prod_{i=1}^4 \frac{d^2 k_i}{(2\pi)^2 k_i^2} \frac{L_\mu(p, k_1)L^\mu(p, k_2)L_\nu(q, k_3)L^\nu(q, k_4)}{(p - k_1)^2(p - k_2)^2(q - k_3)^2(q - k_4)^2} \times \\ & \quad \frac{2(2\pi)^8}{\pi^2 \kappa_4} \left[\int dz^- \tilde{\mu}^2(z^-) \right]^2 \left[\int dz^- \tilde{\mu}^4(z^-) \right]^2 \delta^{ac}\delta^{bd}\delta(k_1 - k_2)\delta(k_3 - k_4) \times \\ & \quad \left(\delta^{a'b'}\delta^{c'd'} + \delta^{a'c'}\delta^{b'd'} + \delta^{a'd'}\delta^{b'c'} \right) \delta(k_1 + k_3 - k_2 - k_4). \end{aligned}$$

The color factor evaluates to

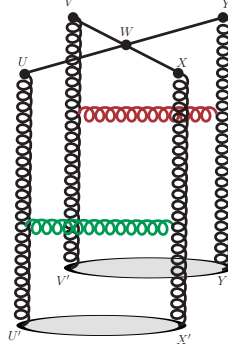


FIG. 5: Connected diagram for two-gluon production from the quartic operator in the action [17].

$$f_{gaa'} f_{g'bb'} f_{gcc'} f_{g'dd'} \delta^{ac} \delta^{bd} \left(\delta^{a'b'} \delta^{c'd'} + \delta^{a'c'} \delta^{b'd'} + \delta^{a'd'} \delta^{b'c'} \right) = 2N_c^2(N_c^2 - 1) + N_c^2(N_c^2 - 1)^2.$$

Using eq. (A2) we get:

$$-\frac{16g^{12}}{(2\pi)^8 \pi^2 \kappa_4} \left[\int dz^- \tilde{\mu}^2(z^-) \right]^2 \left[\int dz^- \tilde{\mu}^4(z^-) \right]^2 [2N_c^2(N_c^2 - 1) + N_c^2(N_c^2 - 1)^2] \times \\ \frac{S_\perp}{p^2 q^2} \int \frac{d^2 k_1}{k_1^2 (p - k_1)^2} \int \frac{d^2 k_3}{k_3^2 (q - k_3)^2}.$$

The integral over the ladder momentum is again cut off at the saturation scale Q_s :

$$\int \frac{d^2 k_1}{k_1^2 (p - k_1)^2} \approx \frac{2\pi}{p^2} \log \frac{p}{Q_s}.$$

Then, the quartic contribution to connected two gluon production becomes

$$-\frac{g^{12}}{4\pi^8 \kappa_4} \left[\int dz^- \tilde{\mu}^2(z^-) \right]^2 \left[\int dz^- \tilde{\mu}^4(z^-) \right]^2 [2N_c^2(N_c^2 - 1) + N_c^2(N_c^2 - 1)^2] \frac{S_\perp}{p^4 q^4} \log \frac{p}{Q_s} \log \frac{q}{Q_s}. \quad (\text{A4})$$

The last step is to express the fully connected diagrams in terms of the single inclusive cross section:

$$\left\langle \frac{dN}{dy} \right\rangle = \frac{g^6}{4\pi^4} \left[\int dz^- \tilde{\mu}^2(z^-) \right]^2 N_c(N_c^2 - 1) \frac{S_\perp}{p^4} \log \frac{p}{Q_s}. \quad (\text{A5})$$

Summing eq. (A3) and eq. (A4) and using eq. (A5) we get:

$$\left\langle \frac{d^2 N}{dy_1 dy_2} \right\rangle_{conn.} = \left[\frac{2\pi}{Q_s^2(N_c^2 - 1)S_\perp} - \frac{4(N_c^2 + 1)}{\kappa_4 S_\perp (N_c^2 - 1)} \frac{[\int dz^- \tilde{\mu}^4(z^-)]^2}{[\int dz^- \tilde{\mu}^2(z^-)]^2} \right] \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle.$$

The fluctuation parameter k^{-1} is now identified with the expression in the square brackets. We rewrite it in terms of

$$\beta \equiv \frac{C_F^2}{6\pi^3} \frac{g^8}{Q_s^2 \kappa_4} \left[\int dz^- \tilde{\mu}^4(z^-) \right]^2, \quad (\text{A6})$$

and use

$$Q_s^2 = \frac{g^4 C_F}{2\pi} \int dz^- \tilde{\mu}^2(z^-), \quad (\text{A7})$$

to arrive at the final expression

$$\frac{N_c^2 - 1}{2\pi} Q_s^2 S_\perp \frac{1}{k} = 1 - 3\beta(N_c^2 + 1). \quad (\text{A8})$$

Appendix B: The moment C_3 with the quartic action

In this section we are going to calculate the connected diagrams for three-gluon production to obtain the correction to the reduced moment C_3 at order $1/\kappa_4 \sim 1/[g(gA^{1/3})^3]$, assuming as before that $gA^{1/3} > 1$. At the end of this section we also outline corrections suppressed by higher powers of $gA^{1/3}$.

We are looking for the contribution of the connected diagrams to the following expression [23]:

$$\begin{aligned} \left\langle \frac{d^3 N}{dy_1 dy_2 dy_3} \right\rangle &= \frac{(-ig^3)^6}{8(2\pi)^9} f_{gaa'} f_{g'bb'} f_{g''cc'} f_{gff'} f_{g'ee'} f_{g''dd'} \times \\ &\int \prod_{i=1}^6 \frac{d^2 k_i}{(2\pi)^2 k_i^2} \frac{L_\alpha(p, k_1) L^\alpha(p, k_2) L_\beta(q, k_3) L^\beta(q, k_4) L_\gamma(l, k_5) L^\gamma(l, k_6)}{(p-k_1)^2 (p-k_2)^2 (q-k_3)^2 (q-k_4)^2 (l-k_5)^2 (l-k_6)^2} \times \\ &\langle \rho_1^{*f}(p-k_2) \rho_1^{*e}(q-k_4) \rho_1^{*d}(l-k_6) \rho_1^a(p-k_1) \rho_1^b(q-k_3) \rho_1^c(l-k_5) \rangle \times \\ &\langle \rho_2^{*f'}(k_2) \rho_2^{*e'}(k_4) \rho_2^{*d'}(k_6) \rho_2^{a'}(k_1) \rho_2^{b'}(k_3) \rho_2^{c'}(k_5) \rangle . \end{aligned} \quad (B1)$$

As before, the ρ correlators of the target and the projectile consist of two parts, one from the quadratic operator in the action and another from the additional ρ^4 operator:

$$\langle \rho^{*f} \rho^{*e} \rho^{*d} \rho^a \rho^b \rho^c \rangle = \langle \rho^{*f} \rho^{*e} \rho^{*d} \rho^a \rho^b \rho^c \rangle_{\text{Gaussian}} + \langle \rho^{*f} \rho^{*e} \rho^{*d} \rho^a \rho^b \rho^c \rangle_{\text{Correction}} .$$

The product of the two Gaussian contributions from the target and the projectile, to leading order in the gluon momenta, gives rise to 16 "rainbow" diagrams. The result has been obtained previously [13] and reads (expressed in terms of the mean multiplicity):

$$\left\langle \frac{d^3 N}{dy_1 dy_2 dy_3} \right\rangle_{\text{Conn. Gaussian}} = \frac{8\pi^2}{Q_s^4 S_\perp^2 (N_c^2 - 1)^2} \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle \left\langle \frac{dN}{dy_3} \right\rangle . \quad (B2)$$

The correction, to first order in κ_4^{-1} is

$$\sim 2 \langle \rho^{*f'} \rho^{*e'} \rho^{*d'} \rho^{a'} \rho^{b'} \rho^{c'} \rangle_{\text{Gaussian}} \langle \rho^{*f} \rho^{*e} \rho^{*d} \rho^a \rho^b \rho^c \rangle_{\text{Correction}} . \quad (B3)$$

Again, we are considering only rainbow diagrams, so for the Gaussian six-point function in the above expression, from all possible contractions, we keep only the term

$$(2\pi)^6 \left[\int dz^- \tilde{\mu}^2(z^-) \right]^3 \delta^{a'f'} \delta^{b'e'} \delta^{c'd'} \delta(k_1 - k_2) \delta(k_3 - k_4) \delta(k_5 - k_6) .$$

To calculate the correction to the six-point function to first order in κ_4^{-1} we factorize it into a product of two- and four-point functions. There are fifteen possible factorizations of that kind. Three of them are disconnected diagrams and the remaining twelve give identical contributions. We consider, for example, the following combination:

$$\begin{aligned} &\langle \rho_1^{*f}(p-k_2) \rho_1^{*e}(q-k_4) \rho_1^{*d}(l-k_6) \rho_1^a(p-k_1) \rho_1^b(q-k_3) \rho_1^c(l-k_5) \rangle \\ &= \langle \rho_1^a(p-k_1) \rho_1^b(q-k_3) \rangle \langle \rho_1^{*f}(p-k_2) \rho_1^{*e}(q-k_4) \rho_1^{*d}(l-k_6) \rho_1^c(l-k_5) \rangle . \end{aligned}$$

The two point function is

$$\langle \rho_1^a(p-k_1) \rho_1^b(q-k_3) \rangle = (2\pi)^2 \left[\int dz^- \tilde{\mu}^2(z^-) \right] \delta^{ab} \delta(p+q-k_1-k_3) ,$$

and for the correction to the four-point function we use the last line from eq. (A3).

The color factor is

$$\begin{aligned} &f_{gaa'} f_{g'bb'} f_{g''cc'} f_{gff'} f_{g'ee'} f_{g''dd'} \delta^{ab} (\delta^{cd} \delta^{ef} + \delta^{ce} \delta^{df} + \delta^{cf} \delta^{de}) \delta^{a'f'} \delta^{b'e'} \delta^{c'd'} \\ &= 2N_c^3 (N_c^2 - 1) + N_c^3 (N_c^2 - 1)^2 . \end{aligned}$$

Putting everything together into eq. (B1) and multiplying by two [because of (B3)] and by twelve (which is the number of possible diagrams) we get:

$$\begin{aligned} \left\langle \frac{d^3 N}{dy_1 dy_2 dy_3} \right\rangle_{\text{Conn. Correction}} &= \frac{48g^{18}\pi}{\kappa_4} \left[\int dz^- \tilde{\mu}^2(z^-) \right]^4 \left[\int dz^- \tilde{\mu}^4(z^-) \right]^2 [2N_c^3(N_c^2 - 1) + N_c^3(N_c^2 - 1)^2] \times \\ &\quad \int \prod_{i=1}^6 \frac{d^2 k_i}{(2\pi)^2 k_i^2} \frac{L_\alpha(p, k_1) L^\alpha(p, k_2) L_\beta(q, k_3) L^\beta(q, k_4) L_\gamma(l, k_5) L^\gamma(l, k_6)}{(p - k_1)^2 (p - k_2)^2 (q - k_3)^2 (q - k_4)^2 (l - k_5)^2 (l - k_6)^2} \times \\ &\quad \delta(k_1 - k_2) \delta(k_3 - k_4) \delta(k_5 - k_6) \delta(p + q - k_1 - k_3) \delta(p + q - k_2 - k_4 - k_6 + k_5) \\ &= -\frac{3g^{18}}{16\kappa_4\pi^{13}} \frac{S_\perp}{p^2 q^2 l^2} \left[\int dz^- \tilde{\mu}^2(z^-) \right]^4 \left[\int dz^- \tilde{\mu}^4(z^-) \right]^2 [2N_c^3(N_c^2 - 1) + N_c^3(N_c^2 - 1)^2] \times \\ &\quad \int \frac{d^2 k_1}{k_1^2 (p + q - k_1)^2 (p - k_1)^4} \int \frac{d^2 k_2}{k_2^2 (l - k_2)^2} . \end{aligned}$$

Again, we regularize the ladder integrals at the saturation scale,

$$\int \frac{d^2 k}{k^2 (p + q - k)^2 (p - k)^4} \simeq \frac{2\pi}{p^2 q^2} \frac{1}{Q_s^2} .$$

Finally, using expression (A5) for the mean multiplicity the ρ^4 contribution to three-gluon production becomes

$$\left\langle \frac{d^3 N}{dy_1 dy_2 dy_3} \right\rangle_{\text{Conn. Correction}} = -\frac{48\pi(N_c^2 + 1)}{\kappa_4 Q_s^2 S_\perp^2 (N_c^2 - 1)^2} \frac{[\int dz^- \tilde{\mu}^4(z^-)]^2}{[\int dz^- \tilde{\mu}^2(z^-)]^2} \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle \left\langle \frac{dN}{dy_3} \right\rangle . \quad (\text{B4})$$

Summing (B2) and (B4),

$$\left\langle \frac{d^3 N}{dy_1 dy_2 dy_3} \right\rangle_{\text{Conn.}} = \left[\frac{8\pi^2}{Q_s^4 S_\perp^2 (N_c^2 - 1)^2} - \frac{48\pi(N_c^2 + 1)}{\kappa_4 Q_s^2 S_\perp^2 (N_c^2 - 1)^2} \frac{[\int dz^- \tilde{\mu}^4(z^-)]^2}{[\int dz^- \tilde{\mu}^2(z^-)]^2} \right] \left\langle \frac{dN}{dy_1} \right\rangle \left\langle \frac{dN}{dy_2} \right\rangle \left\langle \frac{dN}{dy_3} \right\rangle .$$

From the above equation the third reduced moment is:

$$C_3 = \frac{8\pi^2}{Q_s^4 S_\perp^2 (N_c^2 - 1)^2} - \frac{48\pi(N_c^2 + 1)}{\kappa_4 Q_s^2 S_\perp^2 (N_c^2 - 1)^2} \frac{[\int dz^- \tilde{\mu}^4(z^-)]^2}{[\int dz^- \tilde{\mu}^2(z^-)]^2} ,$$

or

$$\frac{(N_c^2 - 1)^2}{4\pi^2} Q_s^4 S_\perp^2 \frac{C_3}{2} = 1 - 9\beta(N_c^2 + 1) , \quad (\text{B5})$$

where we have used expressions (A6) and (A7).

For a NBD we have that $C_3 = 2/k^2$ but if we compare (B5) to the square of eq. (A8), which is

$$\frac{(N_c^2 - 1)^2}{4\pi^2} Q_s^4 S_\perp^2 \frac{1}{k^2} = 1 - 6\beta(N_c^2 + 1) , \quad (\text{B6})$$

we see that the coefficients of the corrections at order $\mathcal{O}(\beta)$ differ. That means that the ρ^4 operator in the action provides a correction to the negative binomial distribution.

In fact, such deviation from a NBD is more obvious if even higher order operators are added to the action. Dropping the longitudinal dependence of the operators for simplicity, such an action would have the form

$$\begin{aligned} S \simeq \int d^2 \mathbf{v}_\perp \left[\frac{\delta^{ab} \rho^a \rho^b}{2\tilde{\mu}^2} - \frac{d^{abc} \rho^a \rho^b \rho^c}{\kappa_3} + \frac{\delta^{ab} \delta^{cd} + \text{perm.}}{\kappa_4} \rho^a \rho^b \rho^c \rho^d - \frac{\delta^{ab} d^{cde} + \text{perm.}}{\kappa_5} \rho^a \rho^b \rho^c \rho^d \rho^e \right. \\ \left. + \frac{(\delta^{ab} \delta^{cd} \delta^{ef} + \text{perm.}) + (d^{abc} d^{def} + \text{perm.})}{\kappa_6} \rho^a \rho^b \rho^c \rho^d \rho^e \rho^f + \dots \right] . \end{aligned}$$

The additional terms are suppressed by powers of $gA^{1/3}$ [24]:

$$\tilde{\mu}^2 \sim g(gA^{1/3}) , \quad \kappa_3 \sim g(gA^{1/3})^2 , \quad \kappa_4 \sim g(gA^{1/3})^3 , \quad \kappa_5 \sim g(gA^{1/3})^4 , \quad \kappa_6 \sim g(gA^{1/3})^5 .$$

The cubic operator gives a correction to the six-point function, i.e. to C_3 at order $1/\kappa_3^2$ but does not correct the four-point function, i.e. $C_2 = 1/k$ (it only renormalizes μ^2). The same applies to the ρ^6 operator: C_3 will contain a term $\sim 1/\kappa_6$ but $1/k$ does not. Hence, beyond a quadratic action the relation $C_3 = 2/k^2$ is not exact.

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